UDC 534

PERIODIC SOLUTIONS OF QUASI-LINEAR SYSTEMS WITH SEVERAL DEGREES OF FREEDOM IN PRESENCE OF ARBITRARY FREQUENCIES

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Methods of constructing periodic solutions of quasi-linear (autonomous or non-autonomous) systems of second order equations, based on the Poicare method, are given in a condensed form. A general case of simple or multiple frequencies of the generating system in the presence of critical (resonant), null and noncritical (nonresonant) frequencies is considered. It is assumed that the amplitude equations have only simple solutions. Two methods of obtaining the coefficients of expansions of the required functions into series in terms of a small parameter are given for the autonomous systems.

Unlike the book by Malkin [1] in which the author considers the systems of first order equations and having obtained the fundamental amplitudes constructs the solutions by successive integration of the equations, the present paper gives for each case a ready formula for computing several approximations.

1. Let us consider a quasi-linear qutonomous system with n degrees of freedom

$$\sum_{k=1}^{n} (a_{ik}x_k^{\cdot\cdot} + c_{ik}x_k) = \mu F_i (x_1,\ldots, x_n, x_1^{\cdot},\ldots, x_n^{\cdot}, \mu)$$
$$a_{ik} = a_{ki}, \quad c_{ik} = c_{ki} \qquad (i, k = 1,\ldots, n) \tag{1.1}$$

Functions $F_i (x_s, x_s^{\cdot}, \mu)$ are assumed to be analytic in x_s and $x_s^{\cdot}$ in their domain of variation, and also in the small parameter μ for the values $0 \leqslant \mu < \mu_0$. We also assume that all roots of the frequency equations of the generating system

$$\Delta (\omega^2) = | c_{ik} - \omega^2 a_{ik} | = 0 \qquad (i, k = 1,\ldots, n) \tag{1.2}$$

are simple and nonnegative. Suppose that these frequencies include, apart from the critical frequencies, the null frequency as well as the noncritical frequencies, e.g.

$$\omega_r = k_r\omega_0 \quad (r = 1,\ldots, l-1), \qquad \omega_l = 0 \tag{1.3}$$

Here k_r are positive integers and ω_0 denotes the frequency of the required periodic solution of the generating system. The noncritical frequencies ω_r are denoted by the subscripts $r = l + 1, \ldots, n$.

Let us construct periodic solutions for the quasi-linear system (1.1) with the period $T = T_0 + \alpha$, where $T_0 = 2\pi / \omega_0$ is the period of the generating solution and α is a function of μ, vanishing at $\mu = 0$.

In [2] it was shown that any solution of a quasi-linear system whose generating system possesses varying frequencies, has the following structure

$$x_k (t) = \sum_{r=1}^{n} p_k^{(r)} x^{(r)} (t), \qquad p_k^{(r)} = \frac{\Delta_{ik} (\omega_r^2)}{\Delta_{i1} (\omega_r^2)} \qquad (i, k = 1,\ldots, n) \tag{1.4}$$

where $\Delta_{ik}(\omega_r^2)$ is the algebraic complement of the element $c_{ik} - \omega_r^2 a_{ik}$ in the determinant $\Delta(\omega_r^2)$ of the formula (1.2). The functions $x^{(r)}(t)$ appearing in the solution (1.4) have the form [3 - 5]

$$x^{(r)}(t) = (A_{r0} + \beta_r)\cos\omega_r t + \frac{B_{r0} + \gamma_r}{\omega_r}\sin\omega_r t +$$
$$+ \sum_{m=1}^{\infty}\left[C_m^{(r)}(t) + \sum_{s=1}^{l}\frac{\partial C_m^{(r)}(t)}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1}\frac{\partial C_m^{(r)}(t)}{\partial B_{s0}}\gamma_s + \ldots\right]\mu^m$$
$$(r = 1, \ldots, l-1, l+1, \ldots n) \qquad (1.5)$$

The function $x^{(l)}(t)$ is obtained from (1.5) by performing a limiting passage with $\omega_l \to 0$

$$x^{(l)}(t) = A_{l0} + \beta_l + (B_{l0} + \gamma_l)t + \sum_{m=1}^{\infty}[C_m^{(l)}(t) + \ldots]\mu^m \qquad (1.6)$$

The initial values of the functions $x^{(r)}(t)$ and $\dot{x}^{(r)}(t)$ are $A_{r0} + \beta_r$ and $B_{r0} + \gamma_r$, respectively. Moreover, β_r and γ_r supplement the initial values of the generating solution and vanish when $\mu = 0$. Using the autonomous property of (1.1) and the condition of periodicity of the generating solution we obtain

$$B_{10} = 0,\ \gamma_1 = 0,\ B_{l0} = 0,\ A_{r0} = B_{r0} = 0\ (r = l+1, \ldots, n) \qquad (1.7)$$

The functions $C_m^{(r)}(t)$ are obtained from the formulas

$$C_m^{(r)}(t) = \Big[\Delta_0\omega_r \prod_{s=1}^{n}{}'(\omega_s^2 - \omega_r^2)\Big]^{-1}\int_0^t R_m^{(r)}(t_1)\sin\omega_r(t - t_1)\,dt_1$$
$$(r = 1, \ldots, l-1, l+1, \ldots, n) \qquad (1.8)$$
$$C_m^{(l)}(t) = \Big[\Delta_0 \prod_{s=1}^{n}{}'\omega_s^2\Big]^{-1}\int_0^t R_m^{(l)}(t_1)(t - t_1)\,dt_1$$

where the prime accompanying the product symbols denotes that the value $s = r$ (or $s = l$) of the index has been omitted. In addition we have

$$\Delta_0 = |a_{ik}| > 0, \qquad R_m^{(r)}(t) = \sum_{i=1}^{n}\Delta_{il}(\omega_r^2)H_{im}(t) \qquad (1.9)$$

The quantities $H_{im}(t)$ represent the coefficients of expansion of the functions $\mu F_i(x_s, \dot{x}_s, \mu)$ with $\beta_s = \gamma_s = 0$ into series in powers of the parameter μ

$$H_{im}(t) = \frac{1}{(m-1)!}\left(\frac{d^{m-1}F_i}{d\mu^{m-1}}\right)_{\beta_s=\gamma_s=\mu=0} \qquad (1.10)$$

The first two of $H_{im}(t)$ in the expanded form are

$$H_{i1}(t) = F_i(x_{10}, \ldots, x_{n0}, \dot{x}_{10}, \ldots, \dot{x}_{n0}, 0) \qquad (i = 1, \ldots, n)$$
$$H_{i2}(t) = \sum_{k=1}^{n}\left(\frac{\partial F_i}{\partial x_k}\right)_0 C_{k1}^{*} + \sum_{k=1}^{n}\left(\frac{\partial F_i}{\partial \dot{x}_k}\right)_0 \dot{C}_{k1}^{*} + \left(\frac{\partial F_i}{\partial \mu}\right)_0 \qquad (1.11)$$

The null subscript accompanying the derivatives in parentheses means that these derivatives are taken at $\beta_s = \gamma_s = \mu = 0$. In general, the functions $C_m^{(r)}(t)$ are not periodic and it can be easily shown that they are analytic functions of the initial values $A_{s0} + \beta_s$ and $B_{s0} + \gamma_s$ only when A_{s0} or B_{s0} are not zero [6]. This explains the choice of the limits of summation with respect to s in (1.5). The conditions of periodicity of the solutions of (1.1) are

$$x^{(r)}(T_0+\alpha)=A_{r0}+\beta_r \ (r=1,\ldots,l),\ x^{(r)}(T_0+\alpha)=\beta_r \ (r=l+1,\ldots,n)$$
$$x^{\cdot(1)}(T_0+\alpha)=0,\ x^{\cdot(r)}(T_0+\alpha)=B_{r0}+\gamma_r \ (r=2,\ldots,l-1)$$
$$x^{\cdot(r)}(T_0+\alpha)=\gamma_r\,(r=l,\ldots,n) \qquad (1.12)$$

We can use one of these conditions, e.g. $x^{(1)}(T_0+\alpha)=0$ to determine the parameter α. We shall seek α in the form of a series in β_s $(s=1,\ldots,l)$, $\gamma_s(2,\ldots,l-1)$ and μ

$$\alpha=\sum_{m=1}^{\infty}\Big[N_m(T_0)+\sum_{s=1}^{l}\frac{\partial N_m}{\partial A_{s0}}\beta_s+\sum_{s=2}^{l-1}\frac{\partial N_m}{\partial B_{s0}}\gamma_s+\ldots\Big]\mu^m \qquad (1.13)$$

Successive differentiation of the equation $x^{(1)}(T_0+\alpha)=0$ with respect to μ yields

$$\left(\frac{\partial\alpha}{\partial\mu}\right)_0=\frac{1}{A_{10}\omega_1^2}C_1^{\cdot(1)}(T_0)=N_1(T_0)$$
$$\left(\frac{\partial^2\alpha}{\partial\mu^2}\right)_0=\frac{2}{A_{10}\omega_1^2}[C_2^{\cdot(1)}(T_0)+N_1C_1^{\cdot\cdot(1)}(T_0)]=2N_2(T_0) \qquad (1.14)$$

etc. Let us now expand the left-hand sides of the conditions of periodicity of the functions $x^{(r)}(t)$ $(r=1,\ldots,l-1)$ and $x^{\cdot(r)}(t)$ $(r=2,\ldots,l)$ in terms of the parameter α and insert into these expressions the value of α given by (1.13). Discarding from the left-hand sides of the resulting expressions the factor $\mu\neq 0$, we obtain

$$\sum_{m=1}^{\infty}\Big[M_{jm}(T_0)+\sum_{s=1}^{l}\frac{\partial M_{jm}}{\partial A_{s0}}\beta_s+\sum_{s=2}^{l-1}\frac{\partial M_{jm}}{\partial B_{s0}}\gamma_s+\ldots\Big]\mu^{m-1}=0$$
$$(j=1,2,\ldots,2l-2) \qquad (1.15)$$

The coefficients $M_{j1}(T_0)$ are

$$M_{r1}(T_0)=C_1^{(r)}(T_0)+N_1B_{r0} \qquad (r=1,\ldots,l-1)$$
$$M_{r+l-2,1}(T_0)=C_1^{\cdot(r)}(T_0)-N_1A_{r0}\omega_r^2 \qquad (r=2,\ldots,l) \qquad (1.16)$$

The coefficients $M_{j2}(T_0)$ accompanying μ are

$$M_{r2}(T_0)=C_2^{(r)}(T_0)+N_1C_1^{(r)}(T_0)+N_2B_{r0}-{}^1/_2N_1^2A_{r0}\omega_r^2 \quad (r=1,\ldots,l-1)$$
$$M_{r+l-2,2}(T_0)=C_2^{\cdot(r)}(T_0)+N_1C_1^{\cdot\cdot(r)}(T_0)-N_2A_{r0}\omega_r^2-{}^1/_2N_1^2B_{r0}\omega_r^2$$
$$(r=2,\ldots,l) \qquad (1.17)$$

etc. In (1.16) and (1.17) we must set $B_{10}=B_{l0}=0$.

Equating the constant terms in the conditions (1.15) to zero and assuming that $\beta_s(0)=\gamma_s(0)=0$, we obtain

$$C_1^{(1)}(T_0)=0,\quad C_1^{(r)}(T_0)+N_1B_{r0}=0$$
$$C_1^{\cdot(r)}(T_0)-N_1A_{r0}\omega_r^2=0,\quad C_1^{\cdot(l)}(T_0)=0 \qquad (r=2,\ldots,l-1) \quad (1.18)$$

The above equations yield the amplitudes $A_{10},\ldots,A_{l0},B_{20},\ldots,B_{l-1,0}$ of the generating solution. In the present paper we shall consider only the cases in which the functional determinant of the emplitude equations differs from zero

$$\Delta^*=\frac{D(M_{11},M_{21},\ldots,M_{2l-2,1})}{D(A_{10},\ldots,A_{l0},B_{20},\ldots,B_{l-1,0})}\neq 0 \qquad (1.19)$$

for the values of the amplitudes obtained, i. e. when the system of equations (1.18) has simple solutions. The parameters β_r and γ_s are expanded into series in integral powers of μ

$$\beta_s = \sum_{m=1}^{\infty} A_{sm}\mu^m \quad (s = 1,\ldots, l), \quad \gamma_s = \sum_{m=1}^{\infty} B_{sm}\mu^m \quad (s = 2,\ldots, l-1) \tag{1.20}$$

Let us substitute these expansions into (1.15) and equate to zero the coefficients of like powers of μ. Since the coefficients accompanying μ in (1.15) are zeros, we obtain the following equations for the coefficients A_{s1} and B_{s1}

$$\sum_{s=1}^{l} \frac{\partial M_{j1}}{\partial A_{s0}} A_{s1} + \sum_{s=2}^{l-1} \frac{\partial M_{j1}}{\partial B_{s0}} B_{s1} + M_{j2}(T_0) = 0 \quad (j = 1,\ldots, 2l-2) \tag{1.21}$$

Similarly we obtain the coefficients A_{s2} and B_{s2}. All equations for A_{sm} and B_{sm} are linear and have the same determinant $\Delta^* \neq 0$.

Inserting the expansions for β_s and γ_s into (1.13) and collecting the terms of like power in μ, we obtain

$$\alpha = T_0 \sum_{m=1}^{\infty} h_m\mu^m, \qquad h_1 = \frac{1}{T_0} N_1(T_0) \tag{1.22}$$

To construct a periodic solution of (1.1) with the period independent of μ and equal to T_0, we perform the following substitution of the independent variable

$$t = \Big(1 + \sum_{m=1}^{\infty} h_m\mu^m\Big)\tau \tag{1.23}$$

When the condition (1.19) holds, all functions $x^{(r)}(\tau)$ can be expanded into a series in integral powers of μ

$$x^{(r)}(\tau) = x_0^{(r)}(\tau) + \mu x_1^{(r)}(\tau) + \ldots \qquad (r = 1,\ldots, n) \tag{1.24}$$

The coefficients of this series are T_0-periodic functions of τ. For the critical frequencies we have

$$x_0^{(r)}(\tau) = A_{r0}\cos\omega_r\tau + \frac{B_{r0}}{\omega_r}\sin\omega_r\tau$$

$$x_1^{(r)}(\tau) = A_{r1}\cos\omega_r\tau + \frac{B_{r1}}{\omega_r}\sin\omega_r\tau + C_1^{(r)}(\tau) + h_1\tau(B_{r0}\cos\omega_r\tau - A_{r0}\omega_r\sin\omega_r\tau) \tag{1.25}$$

$$B_{10} = B_{11} = \ldots = 0 \quad (r = 1, \ldots, l-1)$$

2. We now construct a function $x^{(l)}(t)$ corresponding to the frequency $\omega_l = 0$. Formula (1.6) represents the general form of this function and the value of $C_m^{(l)}(t)$ can be found from the second formula of (1.8).

The process of computing the parameter $\chi = \gamma_l$ represents a specific feature in constructing the function $x^{(l)}(t)$. It can easily be seen that χ is an analytic function of the same parameters as all functions $C_m^{(r)}(t)$ as well as of the parameter μ. Consequently the parameter χ can be written in the form [4]

$$\chi = \sum_{m=1}^{\infty}\Big[S_m + \sum_{s=1}^{l} \frac{\partial S_m}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1} \frac{\partial S_m}{\partial B_{s0}}\gamma_s + \ldots\Big]\mu^m \tag{2.1}$$

Let us substitute this expression into the conditions of periodicity of $x^{(l)}(T_0 + \alpha) = A_{l0} + \beta_l$ and equate to zero the coefficients of like powers of μ. We obtain

$$T_0S_1 + C_1^{(l)}(T_0) = 0, \qquad T_0S_2 + N_1S_1 + C_2^{(l)}(T_0) = 0 \tag{2.2}$$

which will yield, successively, all S_m.

Let us now introduce another function

$$C_m^{(l)*}(t) = C_m^{(l)}(t) + S_m t \tag{2.3}$$

Then we can write the following expression for the function $x^{(l)}(t)$:

$$x^{(l)}(t) = A_{l0} + \beta_l + \sum_{m=1}^{\infty}\left[C_m^{(l)*}(t) + \sum_{s=1}^{l}\frac{\partial C_m^{(l)*}(t)}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1}\frac{\partial C_m^{(l)*}(t)}{\partial B_{s0}}\gamma_s + \ldots\right]\mu^m \tag{2.4}$$

After the substitution $t = h\tau$ the function $x^{(l)}(\tau)$ has the period equal to T_0 and can be expanded into the series (1.24). The first two coefficients of this series are

$$x_0^{(l)}(\tau) = A_{l0}, \qquad x_1^{(l)}(\tau) = A_{l1} + C_1^{(l)*}(\tau) \tag{2.5}$$

3. Finally we construct the functions $x^{(r)}(t)$ corresponding to the noncritical frequencies ω_r $(r = l+1, \ldots, n)$. The method of computing the parameters $\varphi_{r-l} = \beta_r$ and $\psi_{r-l} = \gamma_r$ for $r = l+1, \ldots, n$ represents a specific feature of this process. The parameters φ_{r-l} and ψ_{r-l} are analytic functions of the same quantities as the parameter χ discussed in Sect. 2. We therefore have [5, 6]

$$\varphi_{r-l} = \sum_{m=1}^{\infty}\left[P_m^{(r-l)} + \sum_{s=1}^{l}\frac{\partial P_m^{(r-l)}}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1}\frac{\partial P_m^{(r-l)}}{\partial B_{s0}}\gamma_s + \ldots\right]\mu^m$$
$$(r = l+1, \ldots, n) \tag{3.1}$$

and another analogous expression for the parameter ψ_{r-l} depending on the quantity $Q_m^{(r-l)}$. Let us insert these expressions into the conditions of periodicity (1.12) and equate to zero the coefficients of like powers of μ. This yields two equations for $P_m^{(r-l)}$ and $Q_m^{(r-l)}$.

Let us introduce new functions

$$C_m^{(r)*}(t) = C_m^{(r)}(t) + P_m^{(r-l)}\cos\omega_r t + \frac{Q_m^{(r-l)}}{\omega_r}\sin\omega_r t$$
$$(r = l+1, \ldots, n) \tag{3.2}$$

and the auxilliary functions $W_m^{(r-l)}(t)$ the values of which for $m = 1$ and 2 are

$$W_1^{(r-l)}(t) = C_1^{(r)}(t), \quad W_2^{(r-l)}(t) = C_2^{(r)}(t) + N_1 C_1^{\cdot(r)*}(t) \tag{3.3}$$

Solving the above system of equations we obtain

$$P_m^{(r-l)} = \frac{1}{2}\left[W_m^{(r-l)}(T_0) + \frac{1}{\omega_r}\operatorname{ctg}\frac{\omega_r T_0}{2}\, W_m^{\cdot(r-l)}(T_0)\right]$$
$$Q_m^{(r-l)} = \frac{1}{2}\left[W_m^{\cdot(r-l)}(T_0) - \omega_r \operatorname{ctg}\frac{\omega_r T_0}{2}\, W_m^{(r-l)}(T_0)\right]$$
$$(r = l+1, \ldots, n) \tag{3.4}$$

and we use these formulas to compute the successive values of $P_m^{(r-l)}$ and $Q_m^{(r-l)}$.

The above arguments imply that the functions $x^{(r)}(t)$ for $r = l+1, \ldots, n$ can be written in the form

$$x^{(r)}(t) = \sum_{m=1}^{m}\left[C_m^{(r)*}(t) + \sum_{s=1}^{l}\frac{\partial C_m^{(r)*}(t)}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1}\frac{\partial C_m^{(r)*}(t)}{\partial B_{s0}}\gamma_s + \ldots\right]\mu^m \tag{3.5}$$

After the transformation of the independent variable $t = h\tau$ we find the first two coefficients of the expansion (1.24) which are

$$x_0^{(r)}(\tau) = 0, \quad x_1^{(r)}(\tau) = C_1^{(r)*}(\tau) \qquad (r = l+1, \ldots, n) \tag{3.6}$$

We compute the function $C_{k1}^*(t)$ using the second formula of (1.11) for $H_{i2}(t)$. By the definition of $H_{im}(t)$ given in (1.10) the functions $C_{ku}^*(t)$ should be computed according to the formula

$$C_{ku}^*(t) = \sum_{r=1}^{l-1} p_k^{(r)} C_u^{(r)}(t) + \sum_{r=l}^{n} p_k^{(r)} C_u^{(r)*}(t) \qquad (k = 1, \ldots, n;\ u = 1, \ldots, m-1) \tag{3.7}$$

Here the function $C_u^{(l)*}(t)$ is determined from (2.3) and the functions $C_u^{(r)*}(t)$ for $r = l+1, \ldots, n$ by (3.2). Thus we find that the quantities $H_{im}(t)$ represent a connecting element when the coefficients $x_m^{(r)}(\tau)$ of the expansion (1.24) of the functions $x^{(r)}(\tau)$ are computed from various groups.

4. Expressions for the coefficients of the series (1.24) can be obtained in a different form. To do this, we transform (1.1) to quasi-normal coordinates and make the substitution $t = h\tau$ (1.23). We shall assume h to be a known analytic function of the parameter μ. We have

$$x^{(r)}(t) = z^{(r)}(\tau), \qquad hx^{(r)}(t) = z^{(r)}(\tau) \tag{4.1}$$

As the result of the transformations the system (1.1) becomes [7]

$$z^{(r)''} + \omega_r^2 z^{(r)} = \mu\Phi^{(r)}(z^{(s)}, z^{(s)}, \mu) \tag{4.2}$$

The initial conditions for the functions $z^{(r)}(\tau)$ and their first order derivatives are

$$z^{(r)}(0) = A_{r0} + \beta_r, \qquad z^{(r)}(0) = h(B_{r0} + \gamma_r) \tag{4.3}$$

The autonomous property of the system and the conditions of periodicity of the generating solution yield the results (1.7) obtained previously. The functions $z^{(r)}(\tau)$ can be written in the form

$$z^{(r)}(\tau) = (A_{r0} + \beta_r)\cos\omega_r\tau + \frac{h(B_{r0} + \gamma_r)}{\omega_r}\sin\omega_r\tau +$$
$$+ \sum_{m=1}^{\infty}\left[\bar{C}_m^{(r)}(\tau) + \sum_{s=1}^{l}\frac{\partial\bar{C}_m^{(r)}(\tau)}{\partial A_{s0}}\beta_s + \sum_{s=2}^{l-1}\frac{\partial\bar{C}_m^{(r)}(\tau)}{\partial B_{s0}}\gamma_s + \ldots\right]\mu^m \tag{4.4}$$

When $r = l$ we obtain an expression for $z^{(l)}(\tau)$ analogous to (1.6). Making use of the relations connecting the right-hand sides of the equations written in quasi-normal coordinates $x^{(r)}(t)$ and $z^{(r)}(\tau)$, we can obtain the relations connecting the functions $C_m^{(r)}(\tau)$ and $C_m^{(r)}(\tau)$ directly.

In the new variables the amplitude equations become

$$\bar{C}_1^{(r)}(T_0) = 0 \quad (r = 1, \ldots, l-1), \qquad \bar{C}_1^{(r)}(T_0) = 0 \quad (r = 2, \ldots, l) \tag{4.5}$$

Functions $z^{(r)}(\tau)$ are expanded into series in integral powers of μ

$$z^{(r)}(\tau) = z_0^{(r)}(\tau) + z_1^{(r)}(\tau) + \ldots \qquad (r = 1, \ldots, n) \tag{4.6}$$

Let us bring in new functions $C_m^{(r)*}(\tau)$ for the null frequency, with $r = l$

$$C_m^{(l)*}(\tau) = \bar{C}_m^{(l)}(\tau) + S_m\tau, \qquad S_m = -\frac{1}{T_0}\bar{C}_m^{(l)}(T_0) \tag{4.7}$$

and for the noncritical frequencies with $r = l+1, \ldots, n$

$$\bar{C}_m^{(r)*}(\tau) = \bar{C}_m^{(r)}(\tau) + \bar{P}_m^{(r-l)}\cos\omega_r\tau + \frac{\bar{Q}_m^{(r-l)}}{\omega_r}\sin\omega_r\tau \tag{4.8}$$

Here

$$\bar{P}_m^{(r-l)} = \frac{1}{2}\left[\bar{C}_m^{(r)}(T_0) + \frac{1}{\omega_r}\operatorname{ctg}\frac{\omega_r T_0}{2}\bar{C}_m^{(r)\prime}(T_0)\right]$$

$$(r = l+1, \ldots, n)$$

$$\bar{Q}_m^{(r-l)} = \frac{1}{2}\left[\bar{C}_m^{(r)\prime}(T_0) - \omega_r \operatorname{ctg}\frac{\omega_r T_0}{2}\bar{C}_m^{(r)}(T_0)\right] \qquad (4.9)$$

Then the first two coefficients $z_m^{(r)}(\tau)$ of the series (4.6) are

$$z_0^{(r)}(\tau) = A_{r0}\cos\omega_r\tau + \frac{B_{r0}}{\omega_r}\sin\omega_r\tau$$

$$z_1^{(r)}(\tau) = A_{r1}\cos\omega_r\tau + \frac{B_{r1} + h_1 B_{r0}}{\omega_r}\sin\omega_r\tau + \bar{C}_1^{(r)}(\tau)$$

$$B_{10} = B_{11} = 0 \qquad (r = 1, \ldots, l-1) \qquad (4.10)$$

$$z_0^{(l)}(\tau) = A_{l0}, \qquad z_1^{(l)}(\tau) = A_{l1} + \bar{C}_1^{(l)*}(\tau)$$

$$z_0^{(r)}(\tau) = 0, \qquad z_1^{(r)}(\tau) = \bar{C}_1^{(r)*}(\tau) \qquad (r = l+1, \ldots, n)$$

The following coefficients, e.g. $z_2^{(r)}(\tau)$ are noticeably simpler than $x_2^{(r)}(\tau)$. All terms in the coefficients $z_m^{(r)}(\tau)$ are either constants, or T_0-periodic functions of τ.

5. We consider the case when the frequency equation (1.2) has multiple roots. Suppose that one of these roots has multiplicity d, e.g. $\omega^2 = \omega_1^2 = \ldots = \omega_d^2$.

The presence of multiple frequencies affects only the structure of the solutions of (1.1). In the present case the structure becomes [4]

$$x_k(t) = \sum_{r=1}^{d} q_k^{(r)} x^{(r)}(t) + \sum_{r=d+1}^{n} p_k^{(r)} x^{(r)}(t) \qquad (5.1)$$

The functions $x^{(r)}(t)$ remain unchanged and can be represented by the series (1.24) the coefficients of which are determined, in different cases, by the formulas (1.25), (2.5) and (3.6). As before, the formula (1.4) is used to compute the coefficients $p_k^{(r)}$. The coefficients $q_k^{(r)}$ for $r, \kappa = 1, \ldots, d$ are

$$q_k^{(r)} = 1 \qquad (r = k), \qquad q_k^{(r)} = 0 \qquad (r \neq k) \qquad (5.2)$$

As we know [8], the amplitudes $A_{k0}^{(r)}$ and $B_{k0}^{(r)}$ appearing in the particular solutions of the generating system (1.1) are determined from

$$\sum_{k=1}^{n}(c_{ik} - \omega_r^2 a_{ik}) A_{k0}^{(r)} = 0, \quad \sum_{k=1}^{n}(c_{ik} - \omega_r^2 a_{ik}) B_{k0}^{(r)} = 0 \qquad (5.3)$$

In the case of a multiple root $\omega^2 = \omega_1^2$ of Eq. (1.2) only $n - d$ equations in each of the systems (5.3) are independent, the remaining d equations depend on these $n - d$ equations. Let us arrange the equations in (5.3) so that the first d equations follow from the remaining $n - d$ equations. Solving the set of the last $n - d$ equations for $A_{d+1,0}^{(1)}, \ldots, A_{n0}^{(1)}$ with $i = d+1, \ldots, n$ and $\omega^2 = \omega_1^2$ we obtain

$$A_{k0}^{(1)} = q_k^{(1)} A_{10}^{(1)} + \ldots + q_k^{(d)} A_{d0}^{(1)} \qquad (k = d+1, \ldots, n) \qquad (5.4)$$

and the coefficients $q_k^{(r)}$ $(r = 1, \ldots, d;\ k = d+1, \ldots, n)$ are determined from these relations.

6. Consider finally a quasilinear, nonautonomous system with n degrees of freedom.

$$\sum_{k=1}^{n}(a_{ik}x_k^{\cdot\cdot} + c_{ik}x_k) = f_i(t) + \mu F_i(t, x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, \mu)$$

$$a_{ik} = a_{ki}, \quad c_{ik} = c_{ki} \quad (i = 1, \ldots, n) \tag{6.1}$$

We assume that the functions $F_i(t, x_s, x_s^{\cdot}, \mu)$ are analytic in x_s, x_s and μ, just as in the case of the autonomous systems. Moreover, these functions as well as functions $f_i(t)$ are continuous, 2π-periodic functions of t.

Suppose that the roots of the frequency equation of the generating system (1.2) are simple and nonnegative. Let e.g.

$$\omega_r = k_r \qquad (r = 1, \ldots, l-1), \qquad \omega_l = 0 \tag{6.2}$$

where k_r are positive integers. The frequencies ω_r, $r = l+1, \ldots, n$ are nonresonant.

The necessary condition for the periodic solutions of (6.1) to exist is the absence of the harmonics of order k_r in the functions $f_i(t)$. If the frequencies of the generating system do not include the frequency $k_r = 1$, then periodic solutions of (6.1) can be constructed, with the period $T = 2\pi$. These solutions represent one of the forms of oscillations occurring near the principal resonance. The solutions have the following structure

$$x_k(t) = \varphi_k(t) + \sum_{r=1}^{n} p_k^{(r)} x^{(r)}(t) \tag{6.3}$$

Functions $\varphi_k(t)$ represent a particular solution of the generating system (6.1). The coefficients $p_k^{(r)}$ are determined from the formula (1.4) and the functions $x^{(r)}(t)$ have the form [9]

$$x^{(r)}(t) = (A_{r0} + \beta_r)\cos\omega_r t + \frac{B_{r0} + \gamma_r}{\omega_r}\sin\omega_r t +$$
$$+ \sum_{m=1}^{\infty} C_m^{(r)}(t) + \sum_{s=1}^{l} \frac{\partial C_m^{(r)}(t)}{\partial A_{s0}}\beta_s + \sum_{s=1}^{l-1} \frac{\partial C_m^{(r)}(t)}{\partial B_{s0}}\gamma_s + \ldots\Big]\mu^m \tag{6.4}$$

In contrast to the autonomous system, the summation of the products containing the parameter γ_s raised to various powers is performed in the nonautonomous system from $s = 1$ to $s = l-1$. The passage to the limit as $\omega_l \to 0$ in (6.4) yields the function $x^{(l)}(t)$. The functions $C_m^{(r)}(t)$ are determined from the formulas (1.8) – (1.11) and (3.7).

As the system (6.1) is nonautonomous, the conditions of periodicity of its solutions differ slightly from (1.12) and are

$$x^{(r)}(2\pi) = A_{r0} + \beta_r \quad (r = 1, \ldots, l), \qquad x^{(r)}(2\pi) = \beta_r \quad (r = l+1, \ldots, n)$$
$$x^{\cdot(r)}(2\pi) = B_{r0} + \gamma_r \quad (r = 1, \ldots, l-1), \quad x^{\cdot(r)}(2\pi) = \gamma_r \quad (r = l, \ldots, n). \tag{6.5}$$

They can also be written in the form

$$\sum_{m=1}^{\infty}\Big[C_m^{(r)}(2\pi) + \sum_{s=1}^{l}\frac{\partial C_m^{(r)}}{\partial A_{s0}}\beta_s + \sum_{s=1}^{l-1}\frac{\partial C_m^{(r)}}{\partial B_{s0}}\gamma_s + \ldots\Big]\mu^{m-1} = 0 \tag{6.6}$$
$$(r = 1, \ldots, l-1)$$

together with an analogous formula for the derivative $C_m^{\cdot(r)}(2\pi)$ for $r = 1, \ldots, l$. This yields the following amplitude equations:

$$C_1^{(r)}(2\pi) = 0 \quad (r = 1, \ldots, l-1), \qquad C_1^{\cdot(r)}(2\pi) = 0 \qquad (r = 1, \ldots, l) \tag{6.7}$$

from which we can find the following $2l-1$ amplitudes: $A_{10}, \ldots, A_{l0}$, $B_{10}, \ldots, B_{l-1,0}$.

When the functional determinant of (6.7) is not zero, the parameters β_s $(s = 1, \ldots, l)$ and γ_s $(s = 1, \ldots, l-1)$ are expanded into series in integral powers of μ (1.20). For the coefficients A_{s1} and B_{s1} we have the following equations:

$$\sum_{s=1}^{l} \frac{\partial C_1^{(r)}}{\partial A_{s0}} A_{s1} + \sum_{s=1}^{l-1} \frac{\partial C_1^{(r)}}{\partial B_{s0}} B_{s1} + C_2^{(r)}(2\pi) = 0 \qquad (r = 1, \ldots, l-1) \qquad (6.8)$$

while for $r = 1, \ldots, l$ we have analogous equations in which $C_m^{(r)}(2\pi)$ are replaced by $C_m^{*(r)}(2\pi)$. The functions $x^{(l)}(t)$ and $x^{(r)}(t)$ for $r = l+1, \ldots, n$ are constructed in the same way as in the autonomous system. The parameters χ, φ_{r-l} and ψ_{r-l} are given in the form of expansions analogous to (2.1) and (3.1). Summation of the products containing various powers of the parameters β_s and γ_s is performed in these expansions over the same limits as in those of (6.4), and the remaining formulas are unchanged.

The functions $x^{(r)}(t)$ can be expanded into series in integral powers of μ

$$x^{(r)}(t) = x_0^{(r)}(t) + \mu x_1^{(r)}(t) + \ldots \qquad (r = 1, \ldots, n) \qquad (6.9)$$

Here the coefficients $x_m^{(r)}(t)$ are analogous to the coefficients $z_m^{(r)}(\tau)$ in (4.9) provided that in the latter formulas $C_m^{(r)}(\tau)$ and $\overline{C}_m^{(r)*}(\tau)$ are replaced by $C_m^{(r)}(t)$ and $C_m^{(r)*}(t)$ and that the condition $B_{10} = B_{11} = 0$ is discarded.

When the frequency equation (1.2) has multiple roots $\omega^2 = \omega_1^2 = \ldots = \omega_d^2$ in the case of the nonautonomous system, only the structure of solution is affected and takes the form

$$x_k(t) = \varphi_k(t) + \sum_{r=1}^{d} q_k^{(r)}(x)^{(r)}(t) + \sum_{r=d+1}^{n} p_k^{(r)} x^{(r)}(t) \qquad (6.10)$$

The values of the coefficients $q_k^{(r)}$ are given in Sect. 5.

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